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THE NONHOLONOMIC PROPERTY OF THE ELASTOPLASTIC STATE OF A MEDIUM AND THE CONDITIONS AT STRONG DISCONTINUITIES

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Theory of discontinuities is used to investigate the conditions at a shock wave in an elastoplastic medium with a nonassociative flow rule. A system of relations is proposed at the shock wave which represents, in general, the nonholonomic conditions which become integrable only when the problem of motion of the medium behind the wavefront is solved. In the present case, the Hugoniot adiabate independent of the flow behind the wavefront is absent.

Equations for determining the plastic deformations of materials are generally written in terms of increments and must be integrated when solving specific problems. If the problems are further complicated by the presence of surfaces of strong discontinuities, then the integration can only be performed when the usual equilibrium relations are supplemented by additional boundary conditions at these surfaces. In the present paper we show that, in the absence of the displacement discontinuities, such a condition must be given in the form of the condition of continuity of displacements. The analysis is carried out with the finite character of the deformations taken into account.

The defining incremental constraints are nonholonomic [1] and cannot, in general, be integrated independently. In such cases the relations connecting the parameters of the system at the strong discontinuities cannot be reduced to a system of finite, closed relations. Thus the Hugoniot adiabate will not, in general, exist in dilating plastic materials [2] irrespective of the motion outside the strong discontinuity.

Some authors [3–5] construct additional relations at the strong discontinuity (they can be used to obtain finite relations across the shock) by analyzing the inner structure of the discontinuity, with the help of the same defining equations sometimes supplemented by viscosity terms and a hypothetical loading route. The specific character of the conditions at the strong discontinuity obtained by the passage to the limit from the continuous structure, was noted by Sedov in [1].

In accordance with the approach developed in this paper, we must consider the structure of the shock transition in order to estimate the changes in the initial state (reference state) of the material point passing across the shock front. For this reason the system of equations for the structure must be chosen, in order to be adequate, from the continuous generalized models.

1. Let us consider an elastoplastic medium [6, 7] with a nonassociative flow rule described by linear tensor relationship. We shall use the Eulerian representation of motion.

The total strain e_{ij} increments appear as the sum of their elastic and plastic components

$$De_{ij} = De_{ij}^e + De_{ij}^p = \varepsilon_{ij}Dt \tag{1.1}$$

$$\varepsilon_{ij} = {}^1/_2\,(v_{i,j} + v_{j,i}), \qquad v_{i,j} = \partial v_i \,/\, \partial x_j \tag{1.2}$$

$$\left(\frac{De_{ij}}{Dt} = \frac{\partial e_{ij}}{\partial t} + v_k \frac{\partial e_{ij}}{\partial x_k} - e_{ik}\Omega_{kj} - e_{jk}\Omega_{ki} + e_{ik}\varepsilon_{kj} + e_{jk}\varepsilon_{ki}\right)$$

Here D denote the increments corresponding to the derivative in the Oldroyd's sense [8] (the expression in the parentheses) and ε_{ij} is the strain rate tensor. On the other hand, the total strains e_{ij} themselves which are, generally speaking, not small, are given in terms of the displacements u_{ij} by the following Almansi relation:

$$e_{ij} = {}^1/_2(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \tag{1.3}$$

As we know (see e.g. [8, 9]), the relations (1.1) and (1.3) do not contradict each other since the tensors e_{ij} and ε_{ij} are connected by the formula

$$\varepsilon_{ij} = De_{ij} \,/\, Dt = d'e_{ij} \,/\, dt + e_{ik}\varepsilon_{kj} + e_{jk}\varepsilon_{ki} \tag{1.4}$$

here $d' \,/\, dt$ denotes the derivative in the Jaumann sense.

The small elastic strains De_{ij}^e are connected with the stress increments $D\sigma_{ij}$ by means of the Hooke's law

$$D\sigma_{ij} = \lambda De_{kk}^e\delta_{ij} + 2\mu De_{ij}^e$$

where λ and μ are the Lamé coefficients. The above formula corresponds to the linear elastic law in the Lagrangian coordinates. In terms of the rates of change of the stresses and strains, these relations assume the form

$$\frac{D\sigma_{ij}}{Dt} = \lambda \frac{De_{kk}}{Dt}\,\delta_{ij} + 2\mu\,\frac{De_{ij}}{Dt} - \lambda\frac{De_{kk}^p}{Dt}\,\delta_{ij} - 2\mu\,\frac{De_{ij}^p}{Dt} \tag{1.5}$$

The authors of [6, 7] used the following Jaumann derivatives of the stresses

$$d'\sigma_{ij} \,/\, dt = d\sigma_{ij} \,/\, dt - \sigma_{ik}\Omega_{kj} - \sigma_{jk}\Omega_{ki}$$

and assumed that $\varepsilon_{ij} = de_{ij} / dt$, where $d / dt = \partial \,/\, \partial t + v_k\partial \,/\, \partial x_k$ is a substantive derivative. We note that the relation (1.5) with $e_{ij}^p = 0$ corresponds to the Truesdell [8] model of a linearly hyperelastic material.

In accordance with the principles of the incremental theory of plasticity in the tensor-linear isotropic case, the relation between the plastic strain increments and the stresses can be written in the form [7]

$$De_{ij}^p = d\zeta\,({}^1/_3\sigma_{kk} - H)\delta_{ij} + d\psi\,(\sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}) \tag{1.6}$$

Here ζ, ψ and H are scalar functions of the state parameters of the medium. The functions ζ and ψ are assumed unknown, and to find them we introduce [6] two additional conditions of plastic strain

$$\Phi_\sigma = (\sigma_{ij},\ \alpha,\ Y,\ \chi,\ \ldots) = \sqrt{J_2'} + {}^1/_3\alpha J_1 - Y = 0, \tag{1.7}$$

$$J_1 = \sigma_{kk}, \qquad J_2 = \sigma_{ij}'\sigma_{ij}'$$

$$\Phi_\varepsilon = (\varepsilon_{ij}{}^p,\ \Lambda,\ \chi, \ldots) = I_1 - 2\Lambda\sqrt{I_2'} = 0,\ I_1 = \varepsilon_{kk}^p, \qquad (1.8)$$
$$I_2 = \varepsilon_{ij}{}'^p \varepsilon_{ij}{}'^p$$

The first of these conditions represents the von Mises-Schleicher flow condition for the stresses, and the second condition is a kinematic constraint for the plastic strain rate tensor (the dilatation condition [7]). J_1 and I_1 are the first invariants of the stress tensor and plastic strain rate tensor, J_2 and I_2 are the second invariants of the deviators of the stress and the plastic strain rate tensors, $\alpha(J,\ \chi)$ and $Y(\chi)$ are the coefficients of internal friction and cohesion, χ is the hardening parameter and $\Lambda(\chi,\ I)$ is the dilatation rate.

The dilatation condition (1.8) makes possible the elimination of H and ζ using the following relations: $d\zeta = -{}^2/_3\,\Lambda\alpha d\psi$, $H = \alpha^{-1}Y$. Then the relation between the strains and stresses (1.5) assumes the form

$$De_{kk}{}^p = {}^2/_3\,\Lambda\alpha({}^1/_3\alpha^{-1}Y - \sigma_{kk})d\psi, \qquad De_{ij}{}'^p = \sigma_{ij}{}'d\psi \qquad (1.9)$$

When the law is associative, we have $\alpha \equiv \Lambda$ [7]. The governing equations (1.2), (1.5) and (1.9) and the flow condition (1.7), together with the continuity and motion equations

$$d\rho\ /\ dt + \rho v_{kk} = 0, \qquad \rho dv_i\ /\ dt = \sigma_{ij,j} + F_i \qquad (1.10)$$

form, with the hardening functions $\alpha(J,\ \chi)$ and $Y(\chi)$ known, a closed system of equations of the dynamics of an elastoplastic dilatating medium. Here ρ is the density of the medium and F_i denotes the mass force.

Certain relations in the above system represent finite constraints connecting the basic functions (the condition of flow). Other equations (nonassociative law (1.9) governing the plastic increments, the Hooke's law (1.5) and the possible functional dependence of χ on the process) can only be integrated together with the continuity and motion equations (1.10).

2. Let an isolated surface Σ propagate through the medium and let the stresses, rates of change and their derivatives with respect to the coordinates x_i and time t, suffer a discontinuity at this surface

$$[\sigma_{ij}] \neq 0, \quad [v_i] \neq 0, \ldots, [\partial^k\sigma_{ij}/\partial t^l\partial x_1{}^m \ldots \partial x_2{}^n\partial x_3{}^p] \neq 0$$
$$l + m + n + p = k$$

We assume that the flow conditions hold on at least one side of Σ. The displacements of the material medium are assumed continuous

$$[u_i] = 0 \qquad (2.1)$$

The laws of conservation of mass and impulse at the shock wavefront [1] have the form

$$[\rho G] = 0, \quad [\sigma_{ij}]n_j + \rho G[v_i] = 0 \qquad (2.2)$$

where $G = D - v_n$ is the normal velocity of the shock wave with respect to the material particles, D is the same velocity relative to the fixed coordinate system and n_i is the normal to the surface Σ. We assume that the process of material deformation is not temperature dependent, therefore the balancing law for the energy and the entropy increase at the discontinuity is not given.

In finding the jumps in the values of the finite deformations at the surface of discontinuity Σ we shall use the local n, y_1, y_2 coordinate system which corresponds to the vector n_i and to the vectors τ_{1i} and τ_{2i} tangent to Σ. Because of this, the derivatives in x_i transform to the derivatives in the local coordinates in accordance with the rule

$$\partial / \partial x_i = (\partial / \partial n) n_i + x_{i,\alpha}\, g_{\alpha\beta} \;\; (\partial / \partial y_\beta)$$

$$(x_{i\alpha} = x_{i,\alpha} = \partial x_i / \partial y_\alpha, \;\; g_{\alpha\beta} = x_{i\alpha} x_{i\beta})$$

Here $x_i = x_i(y_1, \; y_2, \; t)$ is the equation of the surface of discontinuity Σ and $g_{\alpha\beta}$ is the metric tensor [10].

Applying the above rule to (1.3) and putting together the differences in the values of the total strains e_{ij} on each side of the surface Σ, we obtain a relation for $[e_{ij}]$. By the condition [10]

$$[\partial u_k / \partial y_\beta] = \partial [u_k] / \partial y_\beta$$

the requirement (2.1) of the continuity of displacement means that

$$[\partial u_k / \partial y_\beta] = 0 \tag{2.3}$$

Therefore the relation for the jump in the values of the total strains is simplified to

$$[e_{ij}] = \frac{1}{2} \left\{ \left[\frac{\partial u_i}{\partial n} \right] n_j + \left[\frac{\partial u_j}{\partial n} \right] n_i - \left[\frac{\partial u_k}{\partial n} \frac{\partial u_k}{\partial n} \right] n_i n_j - \right.$$
$$\left. \left[\frac{\partial u_k}{\partial n} \right] \left(x_{i\alpha} g_{\alpha\beta} \frac{\partial u_k}{\partial y_\beta} n_j + x_{j\alpha} g_{\alpha\beta} \frac{\partial u_k}{\partial y_\beta} n_i \right) \right\}$$

The rate of displacement v_i in the local coordinate system has the form

$$v_i = du_i / dt = \delta u_i / \delta t - G \partial u_i / \partial n + v_k x_{k\alpha} g_{\alpha\beta} du_i / dy_\beta$$

where $\delta / \delta t$ denotes the derivative with respect to time on the moving surface [10]. Again, by virtue of (2.1) we have

$$[\delta u_i / \delta t] = \delta [u_i] / \delta t = 0 \tag{2.4}$$

therefore the jump in the rate of displacement assumes the form

$$[v_i] = - [G \partial u_i / \partial n] + [v_k] x_{k\alpha} g_{\alpha\beta} \partial u_i / \partial y_\beta$$

Passing to the stresses we note that their values on the side of Σ at which plastic deformation occurs, are connected by the plasticity condition. In particular, if we assume that the condition (1.7) holds in front and behind the surface of discontinuity Σ, we can obtain from it the following constraint for the jumps in the values of stresses:

$$[\sqrt{J_2'}] = \sqrt{J_2'^{+}} - \sqrt{J_2'^{+} - 2\sigma_{ij}'^{+} [\sigma_{ij}'] + [\sigma_{ij}'] [\sigma_{ij}']} = -{}^1/_3 [\alpha J_1] + [Y] \tag{2.5}$$

The jumps $[D\sigma_{ij} / Dt]$, $[De_{ij} / Dt]$ and $[De_{ij}^p / Dt]$ are connected by the linear condition (1.5). To express this condition in terms of the jumps in the functions themselves, in their derivatives with respect to the coordinates and of the time derivatives of the jumps in the functions, we first note the following relation:

$$\left[\frac{Da_{ij}}{Dt} \right] = \frac{\delta [a_{ij}]}{\delta t} + \left[v_n \frac{\partial a_{ij}}{\partial n} \right] + \left[v_k x_{k\alpha} g_{\alpha\beta} \frac{\partial a_{ij}}{\partial y_\beta} \right] - D \left[\frac{\partial a_{ij}}{\partial n} \right] -$$
$$\left[\frac{\partial v_k}{\partial n} a_{kj} \right] n_i - \left[\frac{\partial v_k}{\partial y_\alpha} x_{i\beta} g_{\alpha\beta} a_{kj} \right] + \left(\frac{\partial (Dn_k)}{\partial n} n_i + \frac{\partial (Dn_k)}{\partial y_\alpha} x_i{}^\beta g_{\alpha\beta} \right) [a_{kj}] -$$
$$\left[\frac{\partial v_j}{\partial n} n_k a_{ki} \right] - \left[\frac{\partial v_j}{\partial y_\alpha} x_{k\beta} g_{\alpha\beta} a_{ki} \right] + \left(\frac{\partial (Dn_j)}{\partial n} n_k + \frac{\partial (Dn_j)}{\partial y_\alpha} x_{k\beta} g_{\alpha\beta} \right) [a_{ki}]$$

which follows from the definition of the time derivative, in the Oldroyd's sense, of an arbitrary tensor a_{ij}. Setting

$$a_{ij} = \sigma_{ij} - \lambda e_{kk}\delta_{ij} - 2\mu e_{ij} + \lambda e_{kk}{}^{p}\,\delta_{ij} + 2\mu e_{ij}{}^{p}$$

and taking the difference between the expressions (2.3) om the left and right side of Σ, we obtain the following condition on the shock wave:

$$\left[\frac{Da_{ij}}{Dt}\right] = 0 \qquad (2.6)$$

The derivatives of the velocities entering (2.6) can, in turn, be written in terms of the displacements u_i in the following manner:

$$\left[\frac{\partial v_i}{\partial n}\right] = \frac{\delta}{\delta t}\left[\frac{\partial u_i}{\partial n}\right] - \left[G\frac{\partial^2 u_i}{\partial n^2}\right] + \left[v_k x_{k\alpha} g_{\alpha\beta}\frac{\partial}{\partial y_\beta}\frac{\partial u_i}{\partial n}\right]$$

$$\left[\frac{\partial v_i}{\partial y_\alpha}\right] = -\left[G\frac{\partial}{\partial n}\left(x_{i\sigma} g_{\sigma\beta}\frac{\partial u_i}{\partial y_\beta}\right) x_{j\gamma} g_{\gamma\alpha}\right]$$

Finally, the plastic governing relations (1.9) written in the local coordinate system close the equations at the surface of the discontinuity. In fact, setting together the difference between the expressions (1.9) on the left and right side of the front Σ, we obtain

$$\left[\frac{De_{kk}^{p}}{Dt}\right] = \left[\left(\frac{\delta\psi}{\delta t} - G\frac{\partial\psi}{\partial n} + v_k x_{k\beta} g_{\alpha\beta}\frac{\partial\psi}{\partial y_\alpha}\right)\frac{2}{3}\alpha\Lambda\left(\frac{1}{3}Y - \sigma_{kk}\right)\right] \qquad (2.7)$$

$$\left[\frac{De_{ij}'^{p}}{Dt}\right] = \left[\left(\frac{\delta\psi}{\delta t} - G\frac{\partial\psi}{\partial n} + v_k x_{k\beta} g_{\alpha\beta}\frac{\partial\psi}{\partial y_\alpha}\right)\sigma_{ij}'\right]$$

The system (2.1) – (2.7) of relations constitutes the boundary conditions at the surface of discontinuity for the unknown functions σ_{ij}, e_{ij}, $e_{ij}{}^{p}$, v_i, ψ and ρ and makes possible the solution of the Cauchy problem for the system (1.5), (1.7) – (1.10) in the perturbed region, with the boundary conditions on Σ given above. The relations are nonholonomic, since they contain the jumps in the functions themselves and in their normal derivatives, i.e. they cannot, generally speaking, lead to finite relations between the quantities sought at the front, which would be independent of the flow outside the surface of discontinuity.

To integrate the equations of state (1.5) in the region of continuous variation (outside the front), we must know the initial reference states for each material particle, namely the values of the total strains resulting from certain known stresses. These initial states may be the same for all particles (homogeneous initial states), can differ by virtue of the initial inhomogeneity of the material or, which is the most important feature, the shock wave itself can alter these states. For this reason the system of equations at the wavefront given above, must be supplemented by the conditions for the jumps in the initial states.

The magnitudes of these jumps and their dependence on the strength of the shock wave can be found by analyzing the structure of the jump. A generalized continuous model must be employed, and it must be adequate for such passages which cannot be obtained by a continuous motion of the medium in the region under consideration [1].

Any given state of stress $\sigma_{ij}{}^{\circ}$ can be used as the reference state, but the corresponding strains $e_{ij}{}^{\circ}$ and $e_{ij}{}^{p\circ}$ must be known. Then an analysis of the structure is necessary for the determination of the jumps $[e_{ij}^{\circ}]$ and $[e_{ij}^{p\circ}]$ in the material particle of the medium.

Finally, in the most general case, the analysis of the structure of a discontinuous passage within the framework of the generalized models may yield a nonzero jump in the displacements u_i themselves, i.e. the process of specifying the jumps in the initial states and displacements may turn out to be interrelated. The analysis of these variants requires a concretization of the continuous models of the intrashock passages.

3. In the case of a spherically symmetric motion the equations (1.5), (1.7) – (1.10) for the radial stress σ_r and annular stress $\sigma_\theta = \sigma_\varphi$, the radial and annular total and plastic strains e_r, $e_\theta = e_\varphi$, e_r^p, $e_\theta^p = e_\varphi^p$, the displacement u, density ρ and the scalar ψ assume the form

$$\frac{\partial\rho}{\partial t}+v\frac{\partial\rho}{\partial r}+\rho\frac{\partial v}{\partial r}+\rho\frac{v}{r}=0 \tag{3.1}$$

$$\frac{\partial\sigma_r}{\partial t}+\frac{2}{r}(\sigma_r-\sigma_\theta)-\rho\frac{\partial v}{\partial r}-\rho v\frac{\partial v}{\partial r}=0$$

$$\frac{D\sigma_r}{Dt}=\lambda\frac{D}{Dt}(e_r+2e_\theta)+2\mu\frac{De_r}{Dt}-\lambda\frac{D}{Dt}(e_r^p+2e_\theta^p)-2\mu\frac{De_r^p}{Dt}$$

$$\frac{D\sigma_\theta}{Dt}=\lambda\frac{D}{Dt}(e_r+2e_\theta)+2\mu\frac{De_\theta}{Dt}-\lambda\frac{D}{Dt}(e_r^p+2e_\theta^p)-2\mu\frac{De_\theta^p}{Dt}$$

$$\sqrt{6}\varkappa(\sigma_r-\sigma_\theta)=\alpha(\sigma_r+2\sigma_\theta-{}^1/_3Y/\alpha)$$

$$\varkappa=\operatorname{sign}(\sigma_r-\sigma_\theta)=\operatorname{sign}(de_r^p-de_\theta^p)$$

$$e_r=\frac{\partial u}{\partial r}-\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2,\quad e_\theta=\frac{u}{r}-\frac{1}{2}\left(\frac{u}{r}\right)^2$$

$$\frac{D}{Dt}(e_r^p+2e_\theta^p)=\sqrt{\frac{8}{13}}\Lambda\varkappa\frac{D}{Dt}(e_r^p-e_\theta^p)$$

$$\frac{De_\theta^p}{Dt}=\frac{d\psi}{dt}\left\{\frac{2}{3}\Lambda\alpha\left(\frac{1}{3\alpha}Y-\sigma_r-2\sigma_\theta\right)+\frac{1}{3}(\sigma_\theta-\sigma_r)\right\}$$

We can see that the system (3.1) is integrable without the last equation which defines the variation of the function ψ.

The conditions at the surface of discontinuity corresponding to the system (1.1), have the form

$$[\rho G]=0,\qquad [\sigma_r]+\rho G[v]=0 \tag{3.2}$$

$$\frac{\delta[a_{11}]}{\delta t}-\left[G\frac{\partial a_{11}}{\partial r}\right]-2\left[\frac{\partial G}{\partial r}a_{11}\right]=0,\qquad \frac{\delta[a_{22}]}{\delta t}-\left[G\frac{\partial a_{22}}{\partial r}\right]=0$$

$$[e_r]=\left[\frac{\partial u}{\partial r}\right]-\frac{1}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2\right],\qquad [e_\theta]=0$$

$$[v]=-\left[G\frac{\partial u}{\partial r}\right],\qquad \sqrt{6}\varkappa[\sigma_r-\sigma_\theta]=\left[\alpha\left(\sigma_r+2\sigma_\theta-\frac{1}{3\alpha}Y\right)\right]$$

$$\frac{\delta}{\delta t}[e_r^p+2e_\theta^p]-\left[G\frac{\partial}{\partial r}(e_r^p+2e_\theta^p)\right]=$$

$$\sqrt{\frac{8}{3}}\varkappa\left[\Lambda\frac{\delta}{\delta t}(e_r^p-e_\theta^p)-\Lambda G\frac{\partial}{\partial r}(e_r^p-e_\theta^p)\right]$$

$$\frac{\delta' [e_\theta{}^p]}{\delta t} - \left[G \frac{\partial e_\theta{}^p}{\partial r}\right] = \frac{2}{3}\left[\Lambda\alpha\left(\frac{\delta\psi}{\delta t} - G\frac{\partial\psi}{\partial r}\right)\left(\frac{1}{3\alpha}Y - \sigma_r - 2\sigma_\theta\right)\right]$$

where

$$a_{11} = \sigma_r - (\lambda + 2\mu)(e_r - e_r{}^p) - 2\lambda(e_\theta - e_\theta{}^p)$$
$$a_{22} = \sigma_\theta - 2(\lambda + \mu)(e_\theta - e_\theta{}^p) - \lambda(e_r - e_r{}^p)$$

In the case of small strains, the terms

$$[(\partial u / \partial r)^2], \qquad [a_{11}\partial(D - v_2) / \partial r]$$

in the relations at the shock wave can be neglected. For small total strains and a plastically incompressible material, the assumption that the velocity D of propagation of the wave is constant, yields D from the relation $\rho D^2 = (\lambda + {}^2/_3\mu)$ and the value coincides with the velocity of the elastic wave.

In the case of small strains, Eqs. (3.1) and relations (3.2) reduce to those obtained in [11].

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